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CHARACTERIZATIONS OF C^*-ALGEBRAS. II

BY
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1. **Introduction.** The present paper is a continuation of [8], in which two necessary and sufficient conditions were given for a complex Banach algebra with a multiplicative identity element of norm one to be isometrically isomorphic to a C^*-algebra (a norm closed self-adjoint algebra of operators on Hilbert space). In §2 of this paper the principal result of [8] is put into its proper context—the theory of numerical range of operators on a normed linear space. In §3 this theorem is given a simplified proof and several consequences are derived. In particular it is shown that a complex Banach algebra with an identity element of norm one is isometrically isomorphic to a C^*-algebra iff it is linearly isometric. Thus C^*-algebras are a type of Banach algebra which can be described in purely geometric terms without any reference to the multiplication. A proof is also given that the convex hull of the set of exponentials of skew-adjoint elements in a complex C^*-algebra contains the open unit ball. This answers a question raised in [12] where it was shown that the closed convex hull of the set of unitary elements in a C^*-algebra is the closed unti ball. In §4 characterizations are given of those complex Banach spaces which are linearly isometric to a commutative C^*-algebra with identity, or respectively, to an arbitrary C^*-algebra with identity. These results depend on the concept of a vertex of the unit ball in a normed space. The commutative case is related to well-known characterizations of the Banach space of real valued continuous functions on a compact space.

2. **Numerical range.** We begin with a review of the basic facts about numerical range on a Banach space and the application of this concept to Banach algebras. This puts the statement of the main theorem from [8] into its proper context.

Let X be a complex Banach space and let X^* and $[X]$ be, respectively, the Banach space of bounded linear functionals on X, and the Banach algebra of bounded linear operators from X into X. For $x \in X$ we define the conjugate set of x to be $C(x)=\{x^* \in X^* : \|x^*\|=\|x\| \text{ and } x^*(x)=\|x\|^2\}$. The numerical range of $T \in [X]$ is $W(T)=\{x^*Tx : x \in X,\ \|x\|=1;\ x^* \in C(x)\}$. This generalizes the classical concept of numerical range on a Hilbert space. We denote the numerical radius of $T \in [X]$ by $\eta(T)=\sup\{|\lambda| : \lambda \in W(T)\}$ and the spectral radius and spectrum by $\nu(T)$ and $\sigma(T)$, respectively. These concepts have the following basic properties.

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2.1. PROPOSITION. *Let* S, T, $T_\delta \in [X]$ *and* λ, $\mu \in \boldsymbol{C}$. *Then*

(a) $\sigma(T) \subseteq (\text{cl}\,(W(T)) \subseteq$ *closed disk of radius* $\|T\|$.

(b) $W(\lambda S + \mu T) \subseteq \lambda W(S) + \mu W(T)$.

(c) *If* $T_\delta \to T$ *in the weak operator topology,*

$$W(T) \subseteq \lim W(T_\delta) = \{\lambda : \lambda = \lim \lambda_\delta,\ \lambda_\delta \in W(T_\delta)\}.$$

(d) η *is a norm on the linear space* $[X]$ *equivalent to the operator norm and satisfying* $\nu(T) \leqq \eta(T) \leqq \|T\| \leqq e\eta(T)$ (*where* $e = \exp(1)$).

(e) $\sup\{\text{Re}(\lambda) : \lambda \in W(T)\} = \lim_{t \to 0^+} t^{-1}(\|I + tT\| - 1)$.

Proofs of these statements can be found in [6], [7], and [14].

Property (a) shows that $W(T)$ can replace $\sigma(T)$ in certain arguments, and the other properties show why this is an advantage. Note that the numerical radius is not usually an algebra norm (i.e., it fails to satisfy $\eta(ST) \leqq \eta(S)\eta(T)$). Also the numerical range of an operator on a Banach space is not usually convex although this is always true for the classical numerical range on a Hilbert space.

An operator $R \in [X]$ is said to be hermitian iff $W(R) \subseteq \boldsymbol{R}$. Denote the set of hermitian operators in $[X]$ by $[X]_H$.

2.2. PROPOSITION. *The set* $[X]_H$ *is a real linear subspace of* $[X]$ *which is closed in the weak operator topology. If* R, $P \in [X]_H$, *then*

(a) $i(RP - PR) \in [X]_H$;

(b) $\sigma(R) \subseteq \boldsymbol{R}$;

(c) $\max\{\pm t : t \in \sigma(R)\} = \sup\{\pm t : t \in W(R)\} = \pm t^{-1} \log \|e^{\pm tR}\|$ *for all* $t > 0$.

(d) *If* $X = Y^*$, *then* $R \in [X]_H$ *iff* $Rx(y) \in \boldsymbol{R}$ *for all* $y \in Y$ *and all* $x \in C(y)$.

These results are proved in [6], [7], and [13].

In order to apply the numerical range to Banach algebras we consider isometric representations. Since all Banach algebras A in this paper have a unity (a multiplicative identity element of norm one), the left regular representation (i.e., $\Lambda : A \to [A]$, where $\Lambda(S)T = ST$) is isometric. For an algebra with a unity u, we will slightly simplify the notation by considering only representations Υ for which $\Upsilon(u)$ is the identity map.

2.3. PROPOSITION. *Let* A *be a complex Banach algebra with unity* u. *Then for all* $a \in A$ *and all isometric representations* Υ *with* $\Upsilon(u) = I$

$$W(\Lambda(a)) = \{\tau(a) : \tau \in C(u)\} = \text{cl co } W(\Upsilon(a)).$$

Proof. Obviously $\{\tau(a) : \tau \in C(u)\} \subseteq W(\Lambda(a))$. For $b \in A$, $\|b\| = 1$ and $\tau \in C(b)$, define $\tau_b(c) = \tau(cb)$ for all $c \in A$. Then $\|\tau_b\| \leqq 1$, $\tau_b(u) = 1$, so $\tau_b \in C(u)$. Thus $\tau(\Lambda(a)b) = \tau_b(a)$. Since this is a typical element, $W(\Lambda(a)) = \{\tau(a) : \tau \in C(u)\}$.

For $a \in A$ the map $\tau \to \tau(a)$ from A^* into $\boldsymbol{C}$ is a linear map which is continuous when A^* carries the A (i.e., weak*) topology. However, $C(u)$ is convex and compact in the weak* topology so the image $W(\Lambda(a))$ of $C(u)$ under this map is compact and convex.

If Υ is a representation on X and $x \in X$, $\|x\|=1$, $x^* \in C(x)$ then $a \to x^*(\Upsilon(a)x)$ is an element of $C(u)$. Thus cl co $W(\Upsilon(a)) \subseteq W(\Lambda(a))$. On the other hand suppose $\lambda \in W(\Lambda(a))$, $\lambda \notin$ cl co $W(\Upsilon(a))$. By a suitable transformation $a \to \gamma a + \delta$ ($\gamma, \delta \in C$) we may assume that the imaginary axis is the perpendicular bisector of the line segment joining λ to the closest point in cl co $W(\Upsilon(a))$. This contradicts the formula 2.1(e) for the suprema of the real parts of elements in $W(\Upsilon(a))$ and $W(\Lambda(a))$.

2.4. Definition. Let A be a complex Banach algebra with unity. The numerical range and numerical radius of an element $a \in A$ are respectively $W(a) = W(\Lambda(a))$, and $\eta(a) = \eta(\Lambda(a))$.

Most of the results in 2.1 and 2.2 can easily be translated into statements about the numerical range on an algebra. In particular we note that 2.1 (e) implies that $\eta(a) = \sup \{|\lambda|^{-1} \log \|e^{\lambda a}\| : \lambda \in \boldsymbol{C}, \lambda \neq 0\}$ which is the function studied in [1]. Furthermore:

2.5. Proposition. *Let A be a complex Banach algebra with unity u. For $h \in A$ the following are equivalent*:

(a) *The image of h is hermitian in some isometric representation of A.*

(b) *The image of h is hermitian in every isometric representation of A.*

(c) $\tau(h) \in \boldsymbol{R}$ *for all* $\tau \in C(u)$.

(d) $\lim_{t\to 0} t^{-1}(\|u+ith\|-1)=0$, $t \in \boldsymbol{R}$.

(e) $\|\exp(ith)\| = 1$ *for all* $t \in \boldsymbol{R}$.

The set A_H of elements satisfying these conditions is a real linear subspace of A which is closed in the A^* topology and hence in the norm topology. Furthermore $A_H \cap iA_H = \{0\}$.

3. Metric characterizations of C^*-algebras.

We begin by stating the result which provides a starting point for our investigations.

3.1. Theorem [8]. *A complex Banach algebra A with unity is isometrically isomorphic to a C^*-algebra if and only if the linear span of A_H is norm dense in A.*

*In this case each element $a \in A$ has a unique decomposition $a = h + ij$ with h, $j \in A_H$. The map $h + ij \to h - ij$ is an involution on A and any isometric isomorphism of A onto a C^*-algebra is a *-isomorphism relative to this involution.*

The statement of 3.1 in [8] required that $A = A_H + iA_H$. The statement given here is equivalent since the linear span of A_H is $A_H + iA_H$ (because A_H is a real linear subspace) and $A_H + iA_H$ is closed (because the convergence of $\{h_n + ik_n\} \subseteq A_H + iA_H$ implies $\|h_n - h_m\| \leq e\eta(h_n - h_m) \leq e\eta(h_n - h_m + i(k_n - k_m)) \leq e\|h_n + ik_n - (h_m + ik_m)\| \to 0$ so $\{h_n\}$, and similarly $\{k_n\}$, converge to elements in A_H).

We will indicate two slightly different simplified arguments to obtain 3.1 from Lemma 1 of [8]. Theorem 3.1 is then used to strengthen Lemma 1 of [8] so as to answer a question raised in [12]. We also give two simple proofs based on Theorem 3.1 of well-known facts about C^*-algebras. To facilitate the statements of these results we will temporarily call a complex Banach algebra A in which the linear span of A_H is dense a V^*-algebra.

3.2. Lemma. *Let A be a V^*-algebra. Then the map $h+ij \to (h+ij)^* = h-ij$ is a continuous hermitian involution.*

Proof. As shown above $A = A_H + iA_H$ and by 2.5 $A_H \cap iA_H = \{0\}$. Thus (*) is well defined on all of A. It obviously satisfies $(a^*)^* = a$, $(a+b)^* = a^* + b^*$, and $(\lambda a)^* = \bar{\lambda} a^*$ for all $a, b \in A$ and $\lambda \in C$. In order to conclude that $(ab)^* = b^* a^*$ for all $a, b \in A$ it is enough to show that $(hk)^* = kh$ for all $h, k \in A_H$. This follows if $i(hk-kh) \in A_H$ and $hk+kh \in A_H$. The former is essentially 2.2(a) and the latter follows if $j^2 \in A_H$ for each $j \in A_H$. At least $j^2 = h+ik$ for some $h, k \in A_H$. Then $j(h+ik) = (h+ik)j$ so that $jh-hj = i(kj-jk) \in A_H \cap iA_H$ so j commutes with h. Thus $\sigma(j^2) + \sigma(-h) \supseteq \sigma(j^2-h) = i\sigma(k)$. By 2.2(d) $\eta(k) = \nu(k) = 0$ so $k=0$ by 2.1(d). Thus * is an involution. It is hermitian by 2.2(b). Finally $\eta(a) = \eta(a^*)$ so * is continuous by 2.1(d).

3.3. Lemma. *Let A be a V^*-algebra. Then $\nu(h) = \eta(h) = \|h\| = \|h^n\|^{1/n}$ for all $h \in A_H$ and $\|ab\| \leqq \|ba\|$ if $ab \in A_H$.*

Proof. It is enough to consider $h \in A_H$ with $\eta(h) < 1$. Then for $0 < \theta < \pi/2$, $-\sum_{n=1}^{\infty} n^{-1} (\cos(\theta) + i \sin(\theta) h)^n$ converges. If the limit is $k+ij$ with $k, j \in A_H$, then $kj = jk$ and $1 - \cos(\theta) - i \sin(\theta) h = e^{k+ij}$. Thus

$$\begin{aligned} \|(1-\cos(\theta))^2 + (\sin(\theta)^2 h^2)\| &= \|e^{k+ij} e^{k-ij}\| = \|e^{2k}\| = \|e^k\|^2 = \|e^{k-ij}\|^2 \\ &= \|(1-\cos(\theta)) - i(\sin(\theta))h\|^2 \end{aligned}$$

by 2.2(c) and 2.5. Dividing by $(\sin(\theta))^2$ and taking the limit as $\theta \to 0$ shows $\|h^2\| = \|h\|^2$ for all $h \in A_H$ from which the lemma follows.

In any algebra A with an involution denote the set of normal elements by $A_N = \{a \in A : a^* a = aa^*\}$.

3.4. Lemma. *Let A be a V^*-algebra. Then*

(a) $\tau(h)^2 \leqq \tau(h^2)$, $\forall \tau \in C(u)$ *and* $\forall h \in A_H$;

(b) $\|a^* a\| \geqq \eta(a)^2$, $\forall a \in A$;

(c) $\|a^* a\| = \eta(a)^2 = \nu(a)^2$, $\forall a \in A_N$.

Proof. (a) For any real t, $t^2 + 2t\tau(h) + \tau(h^2) = \tau((tu+h)^2) \geqq 0$ by 2.2(c). Thus the discriminant is nonpositive.

(b) Let $a = h+ik$. Then $\|a^* a\| = \nu(a^* a) = \frac{1}{2}(\nu(a^* a) + \nu(aa^*)) = \frac{1}{2}(\eta(a^* a) + \eta(aa^*)) \geqq \eta(h^2+k^2) = \sup\{\tau(h^2+k^2) \in \tau \in C(u)\} \geqq \sup\{\tau(h)^2 + \tau(k)^2 : \tau \in C(u)\} = \eta(a)^2$.

(c) $\|a^* a\| \geqq \eta(a)^2 \geqq \nu(a)^2 = \nu(a^*)\nu(a) \geqq \nu(a^* a) = \|a^* a\|$.

By (c) any commutative *-subalgebra A_0 of A is a C^*-algebra relative to ν as norm. Therefore the usual proof that $\sigma(a^* a)$ in nonnegative for all $a \in A$ ([4, 1.3.10, 1.5, 1.6] or [11]) goes through. Thus the direct sum Π of the representations induced by elements in $C(u)$ ([4], [11]) is faithful and in fact $\|a\|_0 = \|\Pi(a)\|$ is an equivalent norm on A such that $(A, *, \|\cdot\|_0)$ is a C^*-algebra and $\|h\|_0 = \|h\|$ for $h \in A_H$. The next lemma states this result of Vidav [13]. A second proof is also supplied.

3.5. LEMMA. *Let A be a V^*-algebra. Then $\|a\|_0=\|a^*a\|^{1/2}$ (for all $a\in A$) is an equivalent norm on A such that $\|h\|_0=\|h\|$ for all $h\in A_H$ and $(A, \|\cdot\|_0)$ is isometrically isomorphic to a C^*-algebra.*

Proof. By Lemma 3.3 $\|h\|_0=\|h\|$ for $h\in A_H$, and thus $\|aa^*\|_0=\|a^*a\|_0=\|a\|_0^2$ for all $a\in A$. By Lemmas 2.1 and 3.4 we have $e^{-1}\|a\|\leqq\eta(a)\leqq\|a\|_0\leqq(\|a^*\|\ \|a\|)^{1/2}\leqq e^{1/2}\|a\|$. Thus if $\|\cdot\|_0$ is a norm it is equivalent. Clearly $\|\cdot\|_0$ is homogeneous and $\|ab\|_0^2=\|b^*a^*ab\|\leqq\|a^*abb^*\|\leqq\|a^*a\|\ \|bb^*\|=\|a\|_0^2\|b\|_0^2$. Thus it only remains to verify the triangular law. Let $a, b\in A$. Then $\|a+b\|_0^2=\|a^*a+b^*b+a^*b+b^*a\|\leqq\|a^*a\|+\|b^*b\|+\|a^*b+b^*a\|$ so that it is enough to show that $\|a^*b+b^*a\|\leqq 2\|a\|_0\|b\|_0$. For any positive integer m

$$\begin{aligned}\|(a^*b)^m(b^*a)^m\| &\leqq \|(a^*b)^{m-1}(b^*a)^{m-1}\|\ \|b^*aa^*b\|\\ &\leqq \|(a^*b)^{m-1}(b^*a)^{m-1}\|\ \|a\|_0^2\|b\|_0^2.\end{aligned}$$

Thus by induction $\|(a^*b)^m(b^*a)^m\|\leqq\|a\|_0^{2m}\|b\|_0^{2m}$. For any positive integer n,

$$\begin{aligned}&\|(a^*b)^{2^{n-1}}+(b^*a)^{2^{n-1}}\|^2\\ &\quad= \|(a^*b)^{2^n}+(b^*a)^{2^n}+(a^*b)^{2^{n-1}}(b^*a)^{2^{n-1}}+(b^*a)^{2^{n-1}}(a^*b)^{2^{n-1}}\|\\ &\quad\leqq \|(a^*b)^{2^n}+(b^*a)^{2^n}\|+2(\|a\|_0\|b\|_0)^{2^n}.\end{aligned}$$

For any $\varepsilon>0$ there is an integer n such that

$$\begin{aligned}\|(a^*b)^{2^n}\| &\leqq (\nu(a^*b)^2+\varepsilon)^{2^{n-1}} \leqq (\eta(a^*b)^2+\varepsilon)^{2^{n-1}} \leqq (\|b^*aa^*b\|+\varepsilon)^{2^{n-1}}\\ &\leqq (\|a\|_0^2\|b\|_0^2+\varepsilon)^{2^{n-1}}.\end{aligned}$$

Thus

$$\|(a^*b)^{2^n}+(b^*a)^{2^n}\| \leqq 2(\|a\|_0^2\|b\|_0^2+\varepsilon)^{2^{n-1}}.$$

Combining these results

$$\|(a^*b)^{2^k}+(b^*a)^{2^k}\| \leqq 2(\|a\|_0^2\|b\|_0^2+\varepsilon)^{2^{k-1}}$$

for any $1\leqq k\leqq n$. Thus $\|a^*b+b^*a\|^2\leqq 4(\|a\|_0^2\|b\|_0^2+\varepsilon)$ for arbitrary $\varepsilon>0$ and $\|\cdot\|_0$ is a norm.

Finally the Gelfand-Naimark characterization of C^*-algebras shows that $(A, \|\cdot\|_0)$ is a C^*-algebra.

For completeness and later reference we quote the argument from [8] to show that no renorming is actually necessary. Lemma [8]: The closed unit ball of a C^*-algebra A is the closed convex hull of the set $A_e=\{e^{ih} : h\in A_H\}$.

Thus the closed unit ball of $(A, \|\cdot\|_0)$ is contained in the closed unit ball of $(A, \|\cdot\|)$ so $\|a\|\leqq\|a\|_0$ for all $a\in A$. The equality must hold since otherwise

$$\|a^*a\| \leqq \|a^*\|\ \|a\| < \|a^*\|_0\|a\|_0 = \|a^*a\|_0 = \|a^*a\|.$$

All of the statements in 3.1 now follow easily.

Using Theorem 3.1, the lemma from [8] quoted above can be strengthened. This strengthened form answers (at least in part) the question raised in [12] about the nature of the convex hull of the unitary elements in A. It will also be used below.

In the following proposition A_1 and A_1° are the closed and open unit ball in A, respectively. Also

$$A_e = \{\exp(ih) : h \in A_H\} \quad \text{and} \quad A_U = \{v \in A : v^*v = vv^* = u\}.$$

3.6. Proposition. *If A is a complex C^*-algebra with unity then $A_1^\circ \subseteq \operatorname{co} A_e \subseteq A_1$.*

Proof. The expression

$$\|a\|_0 = \inf\left\{\sum_{j=1}^n t_j : a = \sum_{j=1}^n t_j \exp(ih_j),\ h_j \in A_H,\ t_j \geqq 0\right\}$$

is finite for all $a \in A$ and indeed satisfies $\|a\| \leqq \|a\|_0 \leqq 2\|a\|$ since any $h \in A_H \cap A_1^\circ$ can be written as $h = \frac{1}{2}(\exp(ik) + \exp(-ik))$ where $k = \cos^{-1}(h)$ is defined by its power series expansion.

Assume to begin with that A is also commutative. Then $A_e = \{\exp(ik) : k \in A_H\}$ is a group closed under multiplication by complex numbers of norm one. Thus it is easy to check that $\|\cdot\|_0$ is a complex algebra norm for A, and $(A, \|\cdot\|_0)$ is a Banach algebra. This Banach algebra satisfies 3.1 and is thus a C^*-algebra relative to the original involution of $(A, \|\cdot\|)$. For any $a \in A$, $\|a\|_0^2 = \|a^*a\|_0 = \nu(a^*a) = \|a^*a\| = \|a\|^2$. This implies that for any $a \in A_1^\circ$ there are $h_j \in A_H$ and $t_j > 0$ such that $a = \sum_{j=1}^n t_j \exp(ih_j)$ and $\sum_{j=1}^n t_j < 1$. Suitable equal multiples of 1 and -1 can be added to show $a \in \operatorname{co}(A_e)$.

Next we remove the restriction to commutative C^*-algebras. Let A be any C^*-algebra. Apply the commutative case just proved to the commutative C^*-subalgebra generated by a normal element in A. In this way we see that $A_1^\circ \cap A_N \subseteq \operatorname{co}(A_e)$. Since $A_e \subseteq A_U \subseteq (1+\varepsilon)A_1^\circ \cap A_N$ for any $\varepsilon > 0$, the norm $\|\cdot\|_0$ defined above is also given by

$$\|a\|_0 = \inf\left\{\sum_{j=1}^n t_j : a = \sum_{j=1}^n t_j v_j,\ v_j \in A_U,\ t_j \geqq 0\right\}.$$

The same argument as before shows that A is a C^*-algebra relative to this norm which is thus again equal to the original norm. Thus the proposition is proved in the general case.

3.7. Corollary. *In any C^*-algebra*

$$\begin{aligned} \|a\| &= \inf\left\{\sum_{j=1}^n |\lambda_j| : a = \sum_{j=1}^n \lambda_j v_j,\ v_j \in A_U,\ \lambda_j \in \mathbf{C}\right\} \\ &= \inf\left\{\sum_{j=1}^n |\lambda_j| : a = \sum_{j=1}^n \lambda_j \exp(ih_j),\ h_j \in A_H,\ \lambda_j \in \mathbf{C}\right\} \end{aligned}$$

for all $a \in A$.

The first equality was obtained by Russo and Dye [12].

Certain known facts about C^*-algebras can be proved more easily using Theorem 3.1 than using the Gelfand-Naimark characterization. We give two examples.

3.8. Proposition. *Any closed two sided ideal I in a V^*-algebra A is invariant under $(^*)$ and A/I is a V^*-algebra.*

Proof. The expression $\|a+I\|=\inf\{\|a+b\| : b\in I\}$ defines a norm on A/I in which it is a Banach algebra with unity. If $h\in A_H$ then $\|\exp(it(h+I))\|=\|e^{ith}+I\|=1$ for $t\in \mathbf{R}$ so $h+I\in(A/I)_H$. Thus A/I is a V^*-algebra, $A\to A/I$ is a *-homomorphism and I is * invariant.

The next result is a simple proof of a result of P. Civin and B. Yood [2]. The possibility of such a proof was pointed out to the author by F. F. Bonsall.

3.9. Proposition. *If A is a V^*-algebra then A^{**} is a V^*-algebra under the Arens multiplication.*

Proof. Since the involution on A is continuous a continuous involutary map on A^* can be defined by $\tau^*(a)=\tau(a^*)^-$ for all $\tau\in A^*$ and $a\in A$. A similar definition gives an involutary map on A^{**}. Denote the sets of fixed points under these maps by A_H^* and A_H^{**}, respectively. The proof will be complete if $A_H^{**}\subseteq(A^{**})_H$ where A^{**} is the Banach algebra defined by the Arens multiplication. This multiplication is defined by giving an isometric representation Φ of A^{**} on A^* : $\Phi_f(\tau)(a)=f(\Lambda(a)^*(\tau))$ where $f\in A^{**}$, $\tau\in A^*$, $a\in A$, Λ is the left regular representation and $\Lambda(a)^*\in[A^*]$ is the adjoint of $\Lambda(a)\in[A]$. Proposition 2.5 shows that $f\in(A^{**})_H$ iff $\Phi_f\in[A^*]_H$. However by 2.2(d) this is true iff $\Phi_f(\tau)(a)\in\mathbf{R}$ for all $a\in A$ and $\tau\in C(a)$. This is true for $f\in A_H^{**}$ since $\Lambda(a)^*(\tau)\in\|a\|^2C(u)\subseteq A_H^*$.

The conclusion of Proposition 3.9 can be considerably strengthened by use of the next result, which is an immediate consequence of Theorem 3.1 and results on vertices in [1]. This theorem shows that in order to determine whether a Banach algebra with unity is a C^*-algebra only the linear structure and norm need be considered—the multiplication is irrelevant.

3.10. Theorem. *Let A be a Banach algebra with unity. Then A is isometrically isomorphic to a C^*-algebra if and only if it is linearly isometric to a C^*-algebra.*

Proof. We need only prove the sufficiency. Let Φ be a linear isometry of A onto B. The unity u of A is a vertex of the unit ball in A [1]. Thus $\Phi(u)$ is a vertex of the unit ball in B. If B is a C^*-algebra then each vertex is a unitary element [1]. Thus $a\to\Psi(a)=\Phi(u)^*\Phi(a)$ is a linear isometry taking u onto $u\in B$. By 2.6, B is the linear span of B_H. Thus A is the linear span of $\Psi^{-1}(B_H)$. However $\Psi^{-1}(B_H)=A_H$ by 2.5(d). Thus A is isometrically isomorphic to a C^*-algebra by 3.1.

The map Ψ constructed in the proof is known to be a Jordan homomorphism. A new proof of this fact, based on the idea of numerical range, has recently been given by A. L. T. Paterson [9].

Any Banach space can be made into a Banach algebra by defining all products to be zero. Applying this construction to the underlying Banach space of some C^*-algebra, we see that some restriction on A, such as the possession of unity, is necessary in the last theorem. It seems likely that a weaker restriction would suffice. For instance one might merely require that A be semisimple, or that A have an isometric representation.

4. **Banach spaces linearly isometric to C^*-algebras.** Theorem 3.10 suggests the question, "When is a Banach space linearly isometric to some C^*-algebra?" In this section answers will be given to this question, and to the analogous question in the special case where the C^*-algebra is commutative. The commutative case is obviously closely related to the much studied question, "When is a real Banach space linearly isometric to a Banach space $C_{\boldsymbol{R}}(\Omega)$ of all real valued continuous functions on some compact Hausdorff space Ω?" For results on this see [3] and [5]. Both characterizations will involve the concept of a vertex of the unit ball [1].

4.1. Definition. Let X be a complex normed linear space. A vertex (of the unit ball X_1) of X is a point $v \in X$ of norm one for which $C(v)$ separates points in X. Given any $v \in X$ let:

$$\begin{aligned} \eta_v(x) &= \sup\{|\tau(x)| : \tau \in C(v)\} = \text{the } v\text{-radius of } x \in X, \\ X_H^v &= \{x \in X : \tau(x) \in \boldsymbol{R} \text{ for all } \tau \in C(v)\}, \\ X_+^v &= \{x \in X_H^v : \tau(x) \geqq 0 \text{ for all } \tau \in C(v)\}. \end{aligned}$$

A vertex $v \in X$ will be called a normal vertex iff

(a) X is the linear span of X_H^v,

(b) The norm is equal to the v-radius on X.

Clearly the v-radius is a nontrivial norm on X if and and only if $v \neq 0$ and $\|v\|^{-1}v$ is a vertex. If v is a vertex, η_v is a norm satisfying $\eta_v(x) \leqq \|x\|$ for all $x \in X$. If u is the unity of a normed algebra A then the u-radius η_u on A is just the numerical radius, η. Thus the unity in any normed algebra is a vertex. Similarly a unitary element in a C^*-algebra is a vertex. These results were obtained in [1] where it was also shown that every vertex in a C^*-algebra is a unitary element; and that

$$\eta_v(x) = \max_{\theta \in \boldsymbol{R}} \lim_{t \to 0^+} t^{-1}(\|v + te^{i\theta}x\| - \|v\|)$$

for any $v, x \in X$ with $\|v\| = 1$.

Notice that X_H^v and X_+^v are, respectively, a real linear subspace of X and a cone (i.e. $X_+^v + X_+^v \subseteq X_+^v$, $\boldsymbol{R}_+ \cdot X_+^v \subseteq X_+^v$, $X_+^v \cap (-X_+^v) = \{0\}$). They are both closed in the X^* topology and in the η_v topology. Call the order defined on X_H^v by X_+^v the v-order, and assume v is a vertex. Then v is a norm or η_v interior point of $X_+^v \subseteq X_H^v$ and an order unit for the v-order (i.e., for all $x \in X_H^v$ there is a real number t such that $tv \geqq x$). The order interval $[-v, v] = \{x \in X_H^v : -v \leqq x \leqq v\}$ is exactly the η_v closed unit ball of X_H^v. Finally the v-order on X_H^v is Archimedian (i.e., $tv + x \in X_+^v$ for all positive t implies $x \in X_+^v$). If η_v is equivalent to the norm on X then X_+^v is a normal cone by [10, 2.1.7b]. Thus every real linear functional on X_H^v is the difference of two positive linear functionals [10, 2.1.21]. If the norm is equal to the v-radius then the positive linear functionals are exactly the positive multiples of elements of $C(v)$.

In order to connect the vertex with an ordering in X rather than X_H^v, we introduce the following definition.

4.2. Definition. A complex linear space X is said to be normally ordered by a cone X_+ iff there exists an element $v \in X_+$ such that X, X_+, and v satisfy:

(a) X is the complex linear span of X_+.

(b) The real linear span X_H of X_+ satisfies $X_H \cap iX_H = \{0\}$.

(c) For each $x \in X_H$ there is a real number t such that $tv \geqq x$.

(d) If $tv + x \in X_+$ for all $t \in \mathbf{R}_+$ then $x \in X_+$.

If X, X_+, and v satisfy these conditions and $z \in X$ let

$$\mu_v(z) = \max_{\theta \in \mathbf{R}} \min \{t : tv \geqq x \text{ where } x, y \in X_H \text{ and } e^{i\theta}z = x + iy\}.$$

Conditions (c) and (d) show that the minimum in the definition of $\mu_v(z)$ exists. If $z \in X$ and $z = x + iy$ for $x, y \in X_H$ then $e^{i\theta}z = \cos(\theta)x - \sin(\theta)y + i(\cos(\theta)y + \sin(\theta)x)$. This shows that the maximum exists and $\mu_v(z)$ is well defined in $\mathbf{R}_+$. Various linear spaces of complex valued functions are normally ordered by the cone of positive valued functions. For v one chooses the constant function one.

The next theorem shows that the concept of a normally ordered complex linear space and of a normed complex linear space with a normal vertex are essentially equivalent.

4.3. Theorem. *Let* (X, X_+) *be a normally ordered complex linear space and let* v *satisfy* (c) *and* (d) *of the definition. Then* (X, μ_v) *is a complex normed linear space in which* v *is a normal vertex of the unit ball. Moreover*

$$C(v) = \{\tau \in X^* : \tau(X_+) \subseteq \mathbf{R}_+ ; \tau(v) = 1\}, \qquad X_H = X_H^v, \quad \text{and} \quad X_+ = X_+^v.$$

Conversely if X *is a normed complex linear space and* v *is a normal vertex of* X *then* (X, X_+^v) *is a normally ordered complex linear space and* μ_v *is the norm of* X.

Proof. The obvious verification shows that μ_v is a norm. Thus v is a normal vertex if $\eta_v = \mu_v$ and the linear span of X_+^v is X. To show that $\eta_v = \mu_v$ consider $z \in X$, $\theta \in \mathbf{R}$ and $x, y \in X_H$ such that $e^{i\theta}z = x + iy$ and $\mu_v(z) = \min\{t : tv \geqq x\}$. Then $\eta_v(z) = \max_{\varphi \in \mathbf{R}} \lim_{t \to 0^+} t^{-1}(\mu_v(v + te^{i\varphi}z) - 1)$. Clearly the maximum is attained when $\varphi = \theta$ and $\mu_v(v + te^{i\theta}z) = 1 + t\mu_v(z)$. Thus $\eta_v(z) = \mu_v(z)$.

A point $x \in X$ belongs to X_H^v iff $\lim_{t \to 0} t^{-1}(\mu_v(v + itx) - 1) = 0$. However if $x \in X_H$ then

$$e^{i\theta}(v + itx) = \cos(\theta)v - \sin(\theta)tx + i(\sin(\theta)v + \cos(\theta)tx)$$

so $\mu_v(v + itx) \leqq |\cos(\theta)| + |\sin(\theta)|\ |t|\mu_v(x) \leqq (1 + (t\mu_v(x))^2)^{1/2}$. Thus $X_H \subseteq X_H^v$. Since $X_H + iX_H = X$ and $X_H^v \cap iX_H^v = \{0\}$ we see that $X_H^v = X_H$.

Suppose $x \in X_+$ and $\tau \in C(v)$. Then

$$\tau(\mu_v(x)v - x) \leqq \mu_v(\mu_v(x)v - x) = \min\{t : tv \geqq \mu_v(x)v - x\} \leqq \mu_v(x).$$

Thus $\tau(x) = \tau(x - \mu_v(x)v) + \tau(\mu_v(x)v) \geqq -\mu_v(x) + \mu_v(x) = 0$ and $x \in X_+^v$. Thus the linear span of $X_+^v = X_+$ is X and v is a normal vertex.

The last argument shows that $C(v)\subseteq\{\tau\in X^*:\tau(X_+)\subseteq \boldsymbol{R}_+,\ \tau(v)=1\}$. To show the opposite inclusion let τ belong to the set on the right. For $z\in X$ choose $\theta\in\boldsymbol{R}$ so that $|\tau(z)|=\tau(e^{i\theta}z)$ and choose $x, y\in X_H$ such that $e^{i\theta}z=x+iy$. Then $|\tau(z)| =\tau(e^{i\theta}z)=\tau(x)=\tau(\mu_v(x)v)-\tau(\mu_v(x)v-x)\leqq\mu_v(x)\leqq\mu_v(z)$, so that $\tau\in C(v)$.

If v is a normal vertex of a normed linear space X then X, X_+^v, and v are easily seen to satisfy Definition 4.2.

We can now state the main result on Banach spaces which are linearly isometric to a commutative C^*-algebra with unity.

4.4. THEOREM. *Let X be a complex Banach space. Then X is linearly isometric to a commutative C^*-algebra with identity if and only if there is a normal vertex v in X and for any element $x\in X_H^v$ there is an element $y\in X$ such that $(x+X_+^v)\cap X_+^v =y+X_+^v$.*

Proof. Let E be the closure in the X (weak*) topology of the set of extreme points of $C(v)$. Then E is compact in the X-topology. The map $x\to\tilde{x}$, where $\tilde{x}(\tau)=\tau(x)$ for all $x\in X$ and all $\tau\in E$, is an isometric linear map of X into $C(E)$. This map will be onto iff its restriction to X_H^v is onto $C_{\boldsymbol{R}}(E)$. The hypotheses show that X_H^v when ordered by the positive cone X_+^v is an Archimedian ordered lattice with v as order unit. Also the norm of x is μ_v by the last theorem and μ_v when restricted to X_H is just the usual order norm. Thus X_H^v is an Archimedian lattice ordered linear space which is complete in its order norm and thus satisfies Theorem 4.1 of [5] from which we conclude that $x\to\tilde{x}$ is a linear isometry of X onto $C(E)$.

The converse is obvious.

4.5. COROLLARY. *A normally ordered complex linear space X is linearly order isomorphic to a commutative C^*-algebra with identity if and only if X_H is a lattice under the v-order and X is complete with respect to μ_v.*

We turn now to a characterization of noncommutative C^*-algebras as Banach spaces.

4.6. THEOREM. *Let X be a complex Banach space. Then X is linearly isometric to some C^*-algebra with an identity element if and only if there is an element $u\in X$ and a group G of linear transformations on X such that:*

(a) $\|x\| = \inf\{\sum_{j=1}^n |t_j| : x=\sum_{j=1}^n t_jg_j(u),\ t_j\in\boldsymbol{C},\ g_j\in G\}$ *for all* $x\in X$.

(b) $\sum_{j=1}^n t_jg_j(u)=0 \Rightarrow \sum_{j=1}^n t_jg_jg(u)=0$ *for all* g_j, $g\in G$, *and* $t_j\in\boldsymbol{C}$.

(c) $\tau g^{-1}(u)=\tau g(u)^-$ *for all* $\tau\in C(u)$ *and* $g\in G$.

Proof. If X is linearly isometric to a C^*-algebra choose u to correspond to the identity element and G to correspond to the group of left multiplications by unitary elements. The conditions follow easily from previous results. Let X satisfy the given conditions. Then (a) implies that every element $x\in X$ can be expressed as $x=\sum_{j=1}^n t_jg_j(u)$ for suitable $t_j\in\boldsymbol{C}$ and $g_j\in G$. Note that $\|g(x)\|=\|x\|$ for all $g\in G$ and $x\in X$ because of (a). Thus G consists of isometries of X. For each $g\in G$

another map R_g: $X \to X$ can be defined by $R_g : \sum_{j=1}^n t_j g_j(u) = \sum_{j=1}^n t_j g_j g(u)$. This map is well defined by (b) and it is an isometry by (a).

Let A be the linear span of G in $[X]$ with the norm of $[X]$. Obviously A is a normed algebra. Let Φ: $A \to X$ be the linear map $\Phi(\sum_{j=1}^n t_j g_j) = \sum_{j=1}^n t_j g_j(u)$. Since X is the linear span of $G(u)$, Φ is surjective. Notice that $\|u\| \leqq 1$ so that Φ is norm decreasing. Next we show that Φ is injective and in fact norm increasing and hence an isometry. Suppose $\sum_{j=1}^n t_j g_j$ has norm greater than one in A. Then there is some $x \in X$ with $\|x\| < 1 < \|\sum_{j=1}^n t_j g_j(x)\|$. Choose $s_k \in \boldsymbol{C}$ and $h_k \in G$ so that $x = \sum_{k=1}^m s_k h_k(u)$ and $\sum_{k=1}^m |s_k| < 1$. Then

$$1 < \left\| \sum_{j=1}^n t_j g_j(x) \right\| = \left\| \sum_{j=1}^n \sum_{k=1}^m t_j s_k g_j h_k(u) \right\|$$
$$\leqq \sum_{k=1}^m |s_k| \left\| R_{h_k} \sum_{j=1}^n t_j g_j(u) \right\| < \left\| \Phi\left(\sum_{j=1}^n t_j g_j \right) \right\|.$$

Thus Φ is a linear isometry of A onto X and A is a Banach algebra. Notice that Φ maps the identity element of A onto u. Consequently (c) implies that

$$\sum_{j=1}^n (t_j g_j + \bar{t}_j g_j^{-1})$$

is hermitian. Thus finally A is a C^*-algebra since it is the linear span of A_H.

Notice that $g(u)$ is a vertex for each $g \in G$ since u is a vertex and g is an isometry. Thus (a) asserts that the unit ball has a set of vertices so large that its convex hull contains the open unit ball, and that all these vertices are congruent in the sense that there is a transitive group of linear transformations which preserves the set. Condition (b) asserts that this transitive group can be chosen sufficiently small so that a linear combination of its elements which annihilates one vertex in the set annihilates them all.

When X is a C^*-algebra the elements of G may be left or right multiplication by elements in any subgroup of the unitary group which contains the largest connected subgroup of the unitary group.

An interesting class of counterexamples to otherwise plausible variations of Theorem 4.6 can be obtained as follows. Let G be any group and let A be the convolution algebra $l^1(G) = \{f: G \to \boldsymbol{C} : \|f\| = \sum_{\sigma \in G} |f(\sigma)| < \infty\}$. Clearly A is a Banach algebra. For $\sigma \in G$ let $u_\sigma \in A$ be defined by $u_\sigma(\sigma) = 1$, and $u_\sigma(\tau) = 0$ for $\sigma \neq \tau$. Then u_i is a unity for A, where i is the identity element in G, and $A_H = \boldsymbol{R} u_i$. Thus A is not linearly isometric to a C^*-algebra unless G is the trivial group $G = \{i\}$. On the other hand for any finite group G, A satisfies (a) and (b) of 4.6 with $u = u_i$ and $G = \{$left multiplication by $e^{i\theta} u_\sigma : \sigma \in G, \theta \in \boldsymbol{R}\}$. Furthermore for arbitrary G, A is a Banach algebra with an isometric involution $(f^*(\sigma) = f(\sigma^{-1})^-)$, in which the closed unit ball is the closed convex hull of the set of vertices which equals the set

$$\{v \in A : v^{-1} \in A, \|v\| = \|v^{-1}\| = 1\}$$

and is contained in the set $A_U = \{v \in A : vv^* = v^* v = u\}$. Another immediate consequence of 3.1 is relevant to this class of examples.

4.7. COROLLARY. *A Banach algebra A with an involution is isometrically *-isomorphic to a C*-algebra if and only if*

$$\|a\| = \inf\left\{ \sum_{j=1}^{n} |\lambda_j| : a = \sum_{j=1}^{n} \lambda_j u_j,\ \lambda_j \in \boldsymbol{C},\ u_j \in A_U \right\}.$$

Proof. The necessity is just 3.7. To prove the sufficiency first note that the involution is an isometry. Thus for $h=h^* \in A$ and $t \in \boldsymbol{R}$, $\exp(ith) \in A_U$ so $h \in A_H$ and 3.1 applies.

Added in proof. Several topics discussed in this paper are further developed in F. F. Bonsall and J. Duncan, *Numerical range*, London Math. Soc. Lecture Note Series, which will appear shortly. In particular §2 of this paper is similar to F. F. Bonsall, *The numerical range of an element of a normed algebra*, Glasgow Math. J. **10** (1969), 68–72. The two theories were developed independently.

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